# Differential equations

*José Wudka*

April 16, 2004

# 1 The basic problem

- Interactions are believed to be local
- Properties of objects change due to their immediate environment
- **Example(s):**
  - In many mechanical problems the acceleration $\mathbf{a}$ of an object is determined by its surroundings
  - Its surroundings are determined by the objects' position $\mathbf{r}$.
  - There is then a formula

$$\mathbf{a} = \mathbf{f}(\mathbf{r})$$

  - The acceleration is the rate of change of the velocity $\mathbf{a} = \dot{\mathbf{v}}$

– The velocity is the rate of change of position, $\mathbf{v} = \dot{\mathbf{r}}$

– Then $\mathbf{a} = \ddot{\mathbf{r}}$ and

$$\ddot{r}(t) = \mathbf{f}\left(\mathbf{r}(t)\right)$$

which is a differential equation:

A differential equation is a relation between the derivatives of a function and the function itself

– **Example(s):** For a spring that can move only along the $x$ direction, $\mathbf{f}(\mathbf{r}) = -\omega^2 x(r)\hat{\mathbf{x}}$ and the equation becomes

$$(\ddot{x}(t), \ddot{y}(t), \ddot{z}(t)) = \left(-\omega^2 x(t), 0, 0\right)$$

or, in coordinates

$$\ddot{x} = -\omega^2 x \qquad \ddot{y} = 0 \quad \ddot{z} = 0$$

– In other cases the acceleration depends on where the particle is and where it will be shortly thereafter, that is it depends on $\mathbf{v}$ and $\mathbf{r}$: $\mathbf{a} = \mathbf{F}(\mathbf{r}, \mathbf{v})$, or

$$\ddot{\mathbf{r}}(t) = \mathbf{F}\left(\mathbf{r}(t), \dot{\mathbf{r}}(t)\right)$$

- In most *mechanical* problems only **a**, **v** and **r** appear in a differential relation
- In other situations other derivatives of different order can appear.

  **Example(s):**

  – *Radioactive decay:* the change in the number of nuclei $N(t)$ depends linearly on the number of nuclei:

  $$\dot{N} = a + bN$$

  – *Electrostatics in one dimension:* the electric field is the derivative of the electrostatic potential

  $$\mathcal{E} = -\frac{d\phi}{dx}$$

  where $\phi$ is assumed to be known.

- These differential equations have a *parameter*, the solutions are functions of a parameter, the maximum number of derivatives with respect to the parameter is called the order of the equation:

| Equation | parameter | solution | order |
|---|---|---|---|
| $\ddot{x} = -kx$ | $t$ | $x(t)$ | 2 |
| $\mathcal{E} = -d\phi/dx$ | $x$ | $\mathcal{E}(x)$ | 1 |

## 1.1 Partial differential equations

- Sometimes the quantity of interest depends on more than one parameter.
- **Example(s):**
  - The temperature in a room $T$ depends on the position and the time (4 parameters)
  - The density of gas $n$ diffusing through a container depends on position and time
- In this case differential equations involve the *partial* derivatives of the function

- **Example(s):** diffusion in one dimension
  - The change in the concentration at one location $x$, $\partial_t n(x,t)$ depends on how large $n$ is in nearby locations: if $n(x \pm \xi, t) = n(x,t)$ there will be little diffusion.
  - Using a Taylor expansion

    $$n(x \pm \xi, t) = n(x,t) \pm \xi \partial_x n(x,t) + \frac{1}{2}\xi^2 \partial_x^2 n(x,t) + \cdots$$

    the avg. excess particle density near $x$ is then (approx.)

    $$\frac{1}{2}\left[n(x+\xi,t) + n(x-\xi,t)\right] - n(x,t) = \frac{1}{2}\xi^2 \partial_x^2 n(x,t)$$

  - Reasonable to expect that this is proportional to $\partial_t n(x,t)$. The final equation is

    $$\partial_t n(x,t) = C \partial_x^2 n(x,t)$$

- As in the previous case a solution is an explicit function that when substituted into the equation yields and identity.

## 2 The solution

Solving a differential equation implies finding a specific function that identically satisfies the equation when substituted into it

**Important:** in this section I will just give solutions, later I will show one one gets the solutions!

**Example(s):** the harmonic oscillator in 1 dimension

- Differential equation:

$$\ddot{x} = -\omega^2 x$$

- Proposed solution:

$$x = A\cos(\omega t) + B\sin(\omega t)$$

with $A$, $B$ constants. Explicitly,

$$\dot{x} = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$$

$$\ddot{x} = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$$

$$\Rightarrow \qquad \ddot{x} = -\omega^2 x$$

So the proposed solution *is* a solution

- Note that the solution has two undetermined constants $A$ and $B$.

## 2.1 Partial differential equations

Similarly for a partial differential equation a solution is a function of all the independent parameters.

**Example(s):**

$$\left(\partial_t^2 - c^2\partial_x^2\right) f(t,x) = 0$$

where $c$ is a constant; this is called the *wave equation* in 1 spatial dimension.

- A simple solution is

$$f = x - ct$$

- Another simple solution is $f = (x - ct)^2$:

$$\partial_x f \;=\; 2(x - ct)$$

$$
\begin{aligned}
\partial_x^2 f &= 2 \\
\partial_t f &= -2c(x - ct) \\
\partial_t^2 f &= 2c^2
\end{aligned}
$$

$$
\Rightarrow \qquad \left(\partial_t^2 - c^2 \partial_x^2\right) f = 2c^2 - 2c^2 = 0
$$

- In fact $f = (x - ct)^n$ is also a solution
- Moreover any linear combination of solutions is also a solution

A differential equation is *linear* if only linear combination of solutions is also a solution

- The wave equation is linear, therefore

$$f(x,t) = \Sigma\, a_n (x - ct)^n$$

is a solution.

- But $F(u) = \Sigma_n\, a_n u^n$ is an *arbitrary function*. The *general* solution to the wave equation is

$$f(x,t) = F(x - ct)$$

where $F$ is an arbitrary function

The general solution to a *partial* differential equation has one or more arbitrary functions

- The undetermined functions are also fixed by the initial conditions. In this case $t = 0$ we need to specify $f(x, t = 0) = f_0(x)$, then using the general solution for these initial conditions gives

$$F(x) = f_0(x)$$

so that the complete solution for the desired initial conditions is

$$f_0(x - ct)$$

## 3 Finding solutions

Most physics problems are represented in terms of differential equations.

**Example(s):** :

- All of classical mechanics is described in terms of Newton's second law

$$\dot{\mathbf{p}}_a = \mathbf{F}_a\left(\mathbf{r}_1, \ldots \mathbf{r}_N; \mathbf{p}_1, \ldots \mathbf{p}_N\right);\ a = 1, \ldots N$$

$$\mathbf{p}_a = \frac{d(m_a \mathbf{r}_a)}{dt}$$

which are $3N$ coupled, non-linear (in general) ordinary differential equations

- All of electromagnetism is contained in Maxwell's equations

$$\left(\frac{\partial E_x}{\partial x}\right) + \left(\frac{\partial E_y}{\partial y}\right) + \left(\frac{\partial E_z}{\partial z}\right) = \rho$$

$$\left(\frac{\partial B_x}{\partial x}\right) + \left(\frac{\partial B_y}{\partial y}\right) + \left(\frac{\partial B_z}{\partial z}\right) = 0$$

$$\left(\frac{\partial B_y}{\partial z}\right) - \left(\frac{\partial B_z}{\partial y}\right) = \left(\frac{\partial E_x}{\partial t}\right) + J_x \text{ and cyclic perms.}$$

$$\left(\frac{\partial E_y}{\partial z}\right) - \left(\frac{\partial E_z}{\partial y}\right) = -\left(\frac{\partial B_x}{\partial t}\right) \text{ and cyclic prems.}$$

which are 8 linear partial differential equations for theelectric and magnetic fields **E** and **B** in terms of the current **J** and charge density $\rho$.

- The non-relativistic Schrödinger equation for a single particle moving in 3 dimensions

$$-\frac{\hbar^2}{2m}\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)\Psi + V(\mathbf{r})\Psi = i\hbar\Psi$$

which (with its generalization to the case of many particles) described all of microphysics at low energies. It can also be generalized to include all relativistic effects

So, *solving* these differential equations would tell us how things behave. Because of this enormous effort has been devoted to understanding when there are solutions, and how to obtain them. By now there are standard methods.

## 3.1 Equations of the first order

- Simplest form:

$$\dot{x} = ax \quad n' = bn$$

with $a, b$ constants. This is a *first order ordinary differential equation with constant coefficients*

- To solve such equations,

$$\frac{dn}{dx} = bn \quad \Rightarrow \quad \frac{1}{n}\frac{dn}{dx} = b\frac{dx}{dx}$$

$$\frac{d\ln n}{dx} = b\frac{dx}{dx} \quad \Rightarrow \quad \frac{d(\ln n - bx)}{dx} = 0$$

$$\ln n - bx = A = \text{const}$$

$$\Rightarrow \quad n = \exp(A + bx) = e^A e^{bx}$$

- Note that the solution has *one* undetermined constant ($A$)
- Solutions to second-order *ordinary* differential equations in general have *two* undetermined constants
- Solutions to an $l$-th order *ordinary* differential equation (linear or not) in general have $l$ undetermined constants
- The solution with the maximum number of undetermined constants left undetermined is called the *general solution*
- Undetermined constants are fixed by the initial conditions.

**Example(s):** solve $n' = bn$ with $n(x=0) = n_0$

– The *general* solution is

$$n(x) = e^A e^{bx}$$

– At $x = 0$ the general solution equals $n(x=0) = \exp A$.

– For the initial conditions to be satisfied we must have $e^A = n_0$

– The solution with the desired boundary conditions is

$$n = n_0 e^{bx}$$

- Note: not all boundary conditions specify the undetermined constants. For example, solve $n' = bn$ where $b > 0$, with $n(x = -\infty) = 0$
  - The general solution is, as before $n = \exp(A + bx)$
  - As $x \longrightarrow -\infty$ the right hand side $\longrightarrow 0$ *for any* $A$
  - The initial conditions do not specify $A$
- Sometimes it is impossible to satisfy the initial conditions.

**Example(s):** solve $n' = bn$ with $n(x = -\infty) = n_0 \neq 0$

– The general solution is, as before $n = \exp(A + bx)$

– As $x \to -\infty$ the right hand side $\to 0$ *for any $A$*

– One cannot have $n(-\infty) \neq 0$

- One can have systems of equations also, for example

$$\dot{x} = \omega y, \qquad \dot{y} = -\omega x$$

- Often one can turn this into solving a *single* ordinary differential equation, but of a higher order.

**Example(s):**

$$\dot{x} = \omega y, \qquad \dot{y} = -\omega x \quad \Rightarrow$$

$$y = \frac{1}{\omega}\dot{x} \quad \Rightarrow \quad \dot{y} = \frac{1}{\omega}\ddot{x} = -\omega x$$

$$\ddot{x} = -\omega^2 x$$

then one solves the second order equation $\ddot{x} + \omega^2 x$ and then finds $y$ from $y = \dot{x}/\omega$

- Note that the solution to a system of two first-order equations has the same number of undetermined constants as a single second-order equation.
- This is true in general: a system of $l$ first-order equations will have a solution containing $l$ undetermined parameters.

## 3.2 First order equations

All ordinary differential equations are equivalent to a system of first-order equations.

- Start from

$$\frac{d^n f}{dx^n} = \mathcal{F}\left(\frac{d^{n-1} f}{dx^{n-1}}, \frac{d^{n-2} f}{dx^{n-2}}, \ldots \frac{df}{dx}, f\right)$$

- Define

$$y_k = \frac{d^k f}{dx^k}$$

- Then

$$\frac{dy_k}{dx} = \frac{d^{k+1} f}{dx^{k+1}} = y_{k+1}$$

- Then,

$$
\begin{aligned}
\frac{df}{dx} &= y_1 \\
\frac{dy_1}{dx} &= y_2 \\
\frac{dy_2}{dx} &= y_3 \\
\cdots & \cdots \cdots \\
\frac{dy_{n-2}}{dx} &= y_{n-1} \\
\frac{dy_{n-1}}{dx} &= \mathcal{F}(y_{n-1}, y_{n-2}, \ldots, y_1, f)
\end{aligned}
$$

- **Example(s):**

$$\ddot{x} + \omega^2 x = 0$$

for which $n = 2$. I will define $y = \dot{x}$, then

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 x \end{aligned}$$

and can be written in matrix form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

- A generic *linear* ordinary differential equation can also be written in matrix form

$$\frac{d^n f}{dx^n} = a_1 \frac{d^{n-1} f}{dx^{n-1}} + a_2 \frac{d^{n-2} f}{dx^{n-2}} + \cdots + a_{n-1} \frac{df}{dx} + a_n f$$

then if

$$y_k = \frac{d^k f}{dx^k}, \quad y_0 = f$$

I have

$$\frac{d}{dx} \begin{pmatrix} y_{n-1} \\ y_{n-2} \\ \vdots \\ y_2 \\ y_1 \\ y_0 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} & a_n \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_{n-1} \\ y_{n-2} \\ \vdots \\ y_2 \\ y_1 \\ y_0 \end{pmatrix}$$

- The *general form* of an $n$-order linear ordinary differential equation is

$$\frac{d\mathbf{v}}{dx} = \mathbf{M}\mathbf{v}$$

where $\mathbf{v}$ is an $n$-dimensional vector function of $x$ and $\mathbf{M}$ is an *matrix*.

- if **M** happens to be diagonal then it is easy to solve the equation:

$$\frac{d}{dx}\begin{pmatrix} v_n \\ v_{n-1} \\ \vdots \\ v_3 \\ v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} m_n & 0 & \cdots & 0 & 0 & 0 \\ 0 & m_{n-1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & \cdots & m_3 & 0 & 0 \\ 0 & 0 & \cdots & & m_2 & 0 \\ 0 & 0 & \cdots & 0 & & m_1 \end{pmatrix} \begin{pmatrix} v_n \\ v_{n-1} \\ \vdots \\ v_3 \\ v_2 \\ v_1 \end{pmatrix}$$

Then

$$v'_k = m_k v_k$$

with solutions

$$v_k = A_k e^{x m_k}$$

where $A_1, a_2, \ldots, A_n$ are the $n$ undetermined constants to be fixed by some initial conditions.

- Now, many *square* matrices $\mathbf{M}$ can be *diagonalized*:

$$\mathbf{M} = \mathcal{S}^{-1}\mathbf{m}\mathcal{S}$$

where $\mathcal{S}$ is some constant invertible matrix, and $\mathbf{m}$ is a diagonal matrix. Now, let

$$\mathbf{w} = \mathcal{S}^{-1}\mathbf{v} \qquad \Rightarrow \quad \mathbf{v} = \mathcal{S}\mathbf{w}$$

then

$$\frac{d\mathbf{w}}{dx} = \frac{d\mathcal{S}^{-1}\mathbf{v}}{dx} = \mathcal{S}^{-1} \underbrace{\frac{d\mathbf{v}}{dx}}_{=\mathbf{M}\mathbf{v}}$$

$$= \mathcal{S}^{-1}\mathbf{M}\mathbf{v} = \mathcal{S}^{-1}\mathbf{M}\mathcal{S}\mathbf{w} = \mathbf{m}\mathbf{w}$$

Then

$$w_k = A_k e^{x\, m_k},$$

$$v_l = \sum_{k=1}^{N} \mathcal{S}_{lk} w_k = \sum_{k=1}^{N} \mathcal{S}_{lk} A_k e^{x\, m_k}$$

Solving a linear differential equation of the form

$$\dot{\mathbf{v}} = \mathbf{M}\mathbf{v}$$

with $\mathbf{M}$ constant is trivial provided $\mathbf{M}$ can be diagonalized

- **Example(s):**

$$\ddot{x} + \omega^2 x = 0$$

for which $n = 2$. I will define $y = \dot{x}$, then

$$\dot{x} = y \quad \dot{y} = -\omega^2 x$$

and can be written in matrix form

$$\dot{\mathbf{v}} = \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Now diagonalize the matrix. The general form of $\mathcal{S}$ is

$$\mathcal{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \mathcal{S}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

then

$$\mathcal{S}\mathbf{M}\mathcal{S}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$
= \frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -c & a \\ -\omega^2 d & \omega^2 b \end{pmatrix}
$$

$$
= \frac{1}{ad-bc}\begin{pmatrix} -ac-\omega^2 bd & a^2+\omega^2 b^2 \\ -c^2-\omega^2 d^2 & ac+\omega^2 bd \end{pmatrix}
$$

Now, for this to be diagonal,

$$
a^2+\omega^2 b^2 = 0 \quad c^2+\omega^2 d^2 = 0
$$

with solutions

$$
a = \pm i\omega b, \quad c = \pm i\omega d \quad \Rightarrow ad-bc = 0
$$

or

$$
a = \pm i\omega b, \quad c = \mp i\omega d \quad \Rightarrow ad-bc = \pm 2i\omega bd
$$

The first case does not work, so I use the second:

$$\mathcal{S}\mathbf{M}\mathcal{S}^{-1} = \frac{1}{\pm 2i\omega bd}\begin{pmatrix} -2\omega^2 bd & 0 \\ 0 & 2\omega^2 bd \end{pmatrix}$$
$$= \begin{pmatrix} \pm i\omega & 0 \\ 0 & \mp i\omega \end{pmatrix}$$

I can choose either sign (I will use the first) then

$$m_1 = i\omega, \quad m_2 = -i\omega$$

So the solution for the $\mathbf{w}$ is

$$w_1 = A_1 e^{i\omega t} \quad w_2 = A_2 e^{-i\omega t}$$

so that

$$
\begin{aligned}
\mathbf{v} &= \mathcal{S}\mathbf{w} \\
&= \begin{pmatrix} i\omega b & b \\ -i\omega d & d \end{pmatrix} \begin{pmatrix} A_1 \exp(i\omega t) \\ A_2 \exp(-i\omega t) \end{pmatrix} \\
&= \begin{pmatrix} i\omega b A_1 \exp(i\omega t) + bA_2 \exp(-i\omega t) \\ -i\omega c A_1 \exp(i\omega t) + cA_2 \exp(-i\omega t) \end{pmatrix} \\
\Rightarrow \quad x &= v_1 = i\omega b A_1 \exp(i\omega t) + bA_2 \exp(-i\omega t) \\
&= \underbrace{b(i\omega A_1 + A_2)}_{=B'} \cos(\omega t) + \\
&\qquad + \underbrace{b(i\omega A_1 - A_2)}_{=A'} \sin(\omega t) \\
&= A' \sin(\omega t) + B' \cos(\omega t)
\end{aligned}
$$

### 3.3 Partial differential equations

These are harder so solve. I will just talk about one procedure: *separation of variables*.

- For a partial differential equation with parameters $x, y$ try out a solution os the form $X(x)Y(y)$.

- **Example(s):** the equation

$$(\partial_x^2 + \partial_y^2)\phi = 0$$

known as *Laplace's equation in 2 dimensions*.

  – I guess:

$$\phi(x, y) = X(x)Y(y)$$

  and substitute, I get

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{y^2} = 0$$

– Divide by $XY$:

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{y^2}$$

– Since the left hand side is independent of $y$ while the right hand side depends only on $y$ I must have

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{y^2} = C$$

where $C$ is a constant.

– Assume, for example that $C = -k^2$, then

$$\frac{d^2X}{dx^2} = -k^2x \quad \frac{d^2Y}{dy^2} = k^2Y$$

– The solutions are

$$\begin{aligned} X &= A\cos(kx) + B\sin(kx) \\ Y &= ae^{ky} + be^{-ky} \end{aligned}$$

so that the solution becomes

$$\phi_{\text{spec}} = (A\cos(kx) + B\sin(kx))\left(ae^{ky} + be^{-ky}\right)$$

where "spec" indicates this is a special solution.

– But what is $k$? It is completely undetermined! But for each $k$ we have a solution with its constants $A$, $B$ , $a$ and $b$, so these are functions of $k$:

$$\phi_{\text{spec}}(x, y; k) = [A(k)\cos(kx) + B(k)\sin(kx)]\left[a(k)e^{ky} + b(k)e^{-ky}\right]$$

– Since the equation is linear a linear combination of solutions is also a solution

$$u(k_1)\phi_{\mathrm{spec}}(x, y; k_1) + v(k_2)\phi_{\mathrm{spec}}(x, y; k_2)$$

– I can then add an arbitrary number of solutions with arbitrary prefactors, or, since $k$ is a continuous variable,

$$
\begin{aligned}
\phi(x,y) &= \int dk\ u(k)\phi_{\text{spec}}(x,y;k) \\
&= \int dk\ u(k)\left[A(k)\cos(kx)+B(k)\sin(kx)\right]\left[a(k)e^{ky}+b(k)e^{-ky}\right] \\
&= \int dk \left[\tilde{A}(k)\cos(kx)+\tilde{B}(k)\sin(kx)\right]\left[\tilde{a}(k)e^{ky}+\tilde{b}(k)e^{-ky}\right] \\
&= \int dk e^{ky}\left\{\tilde{a}(k)\left[\tilde{A}(k)\cos(kx)+\tilde{B}(k)\sin(kx)\right]\right. \\
&\quad + \qquad \left.\tilde{b}(-k)\left[\tilde{A}(-k)\cos(kx)-\tilde{B}(-k)\sin(kx)\right]\right\} \\
&= \int dk e^{ky}\left[\mathcal{A}(k)\cos(kx)+\mathcal{B}(k)\sin(kx)\right]
\end{aligned}
$$

where $\tilde{A} = uA$, etc. and I used the fact that cos is even in its argument while sin is odd.

This is the *general* solution, it depends on 2 arbitrary functions $\mathcal{A}, \mathcal{B}$.

– Writing out the trigonometric in terms of exponentials it is easy to see that the general solution is of the form

$$\phi = \Phi_+(y + ix) + \Phi_-(y - ix)$$

where $\Phi_\pm$ are arbitrary functions

- **Example(s):** : the equation

$$\left(\partial_t^2 - c^2\partial_x^2\right) f = 0$$

(the wave equation in one space dimension).

  – I guess $f(x,t) = X(x)T(t)$

  – As before this implies

$$\frac{1}{X}X'' = \frac{1}{c^2T}\ddot{T} = -k^2$$

– the solutions are

$$
\begin{aligned}
X &= A\sin(kx) + B\cos(kx) \\
&= A'e^{ikx} + B'e^{-ikx} \\
T &= a'e^{ikct} + b'e^{-ikct} \\
f_{\rm spec} &= A'b'e^{ik(x-ct)} + B'a'e^{-ik(x-ct)} \\
&\quad + A'a'e^{ik(x+ct)} + B'b\text{''}\,e^{-ik(x+ct)}
\end{aligned}
$$

The general solution is ($A'$ and all the other constants are functions of $k$),

$$f(x,t) = \int dk\; u(k) \Big[ A'b'e^{ik(x-ct)} + B'a'e^{-ik(x-ct)}$$

$$+ A'a'e^{ik(x+ct)} + B'b'e^{-ik(x+ct)} \Big]$$

Let

$$F_-(\xi) = \int dk\ u(k) \left[A'b'e^{ik\xi} + B'a'e^{-ik\xi}\right]$$

$$F_+(\xi) = \int dk\ u(k) \left[A'a'e^{ik\xi} + B'b'e^{-ik\xi}\right]$$

then the general solution is

$$f(x,t) = F_-(x - ct) + F_+(x + ct)$$

It can be shown that most decent functions can be represented in the same form as $F_\pm$ so these are *quite arbitrary*.